

**Phys 410**  
**Spring 2013**  
**Lecture #16 Summary**  
**24 October, 2013**

We considered the two-body problem of two objects interacting by means of a conservative central force, with no other external forces acting. This problem eventually simplifies from that of 6 degrees of freedom (for 2 particles in three dimensions) to a single particle moving in one dimension! The Lagrangian can be simplified by adopting the generalized coordinates: relative coordinate  $\vec{r} = \vec{r}_1 - \vec{r}_2$ , and the center of mass coordinate  $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$ , where  $M = m_1 + m_2$  is the total mass. The two-particle Lagrangian simplifies to  $\mathcal{L}(\vec{R}, \vec{r}) = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$ , where  $\mu = m_1m_2/M$  is called the reduced mass because it is smaller than either  $m_1$  or  $m_2$ . Because this Lagrangian is independent of  $\vec{R}$ , it means that the center of mass (CM) momentum  $M\dot{\vec{R}}$  is constant. The other Lagrange equation gives  $\mu\ddot{\vec{r}} = -\vec{\nabla}U(r)$ , which is Newton's second law for the relative coordinate.

Taking advantage of the CM conserved momentum, we can jump to the CM (inertial) reference frame, where the CM is at rest, and the two particles are always moving with equal and opposite momenta. In this reference frame, the Lagrangian simplifies to  $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$ . Because only central forces act, the net torque that the particles exert on each other is zero, hence the total angular momentum of the particles ( $\vec{L}$ ) as seen in this reference frame is conserved. Writing the sum of the angular momenta of the two particles, as seen in the CM reference frame, in terms of the generalized coordinates, we find  $\vec{L} = \vec{r} \times \mu\dot{\vec{r}}$ , which is the same as the angular momentum of a single particle of mass  $\mu$ . Because  $\vec{L}$  is conserved (including its direction), the vectors  $\vec{r}$  and  $\dot{\vec{r}}$  must remain in a fixed two-dimensional plane throughout the motion. This means that the motion is strictly two-dimensional!

Now we have to solve the remaining two-dimensional motion problem with this Lagrangian:  $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$ . Going over to polar coordinates for  $\vec{r}$ , we get  $\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$ . There are two Lagrange equations that follow from this Lagrangian. First we note that  $\phi$  is an ignorable coordinate, hence the angular momentum of the 'particle' is conserved:  $\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{constant}$ . This is in fact just the z-component of the total angular momentum vector  $\vec{L}$  that we calculated above. We give it a new name,  $\ell$ , because it is a constant of the motion (you may now recognize the notation from the quantum treatment of the Hydrogen atom). The other Lagrange equation (for  $r$ ) gives  $\mu\ddot{r} = \mu r \dot{\phi}^2 - dU/dr$ . The first term on the RHS is the centrifugal force for the 'particle'. Solving the angular momentum equation for  $\dot{\phi}$  gives  $\dot{\phi} = \ell/\mu r^2$ , and the radial equation of motion can be

written in terms of  $\ell$  as  $\mu\dot{r} = \ell^2/\mu r^3 - dU/dr$ . The first term on the RHS can be written in terms of a derivative as  $-\frac{d}{dr}(\ell^2/2\mu r^2)$ , so that it can be combined with the potential to create a new “effective potential”  $U_{eff}(r) = U(r) + \ell^2/2\mu r^2$ . The equation of motion finally reduces to a simple one-dimensional form:  $\mu\ddot{r} = -dU_{eff}/dr$ .

Using the example of gravity for  $U(r)$  we found that the effective potential (for  $\ell > 0$ ) has a minimum at a finite value of  $r$ , diverges as  $r$  goes to zero, and approaches zero from below as  $r$  goes to infinity. We found that mechanical energy for the relative coordinate is conserved and equal to  $E = \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} + U(r)$ . Since kinetic energy is either positive or zero, the particle must be located in a region where  $E \geq U_{eff,min}$ . We see that when  $E > 0$  the particle has an un-bounded orbit, while when  $E < 0$  it has a bounded orbit trapped between minimum and maximum values of  $r$ .